SMOOTH MANIFOLDS FALL 2022 - MIDTERM

SOLUTIONS

Problem 1. Let Γ be a countable group acting by homeomorphisms of a topological surface X. Assume that $\varphi : U \to \mathbb{R}^2$ is a chart of X such that

- $\bigcup_{\gamma\in\Gamma} \gamma(U) = X$, and
- for every $\gamma \in \Gamma$, $\varphi \circ \gamma \circ \varphi^{-1} : \varphi(\gamma^{-1}(U) \cap U) \to \varphi(U)$ is C^∞.

Show that there exists a smooth structure on X for which the action of Γ is C^∞.

Solution. We build a C^∞ atlas of charts. It follows that a C^∞ atlas of charts determines a unique smooth structure on X. For each $\gamma \in \Gamma$, let $\varphi_\gamma = \varphi \circ \gamma^{-1} : \gamma(U) \to \mathbb{R}^n$. Since each γ is a homeomorphism and φ is a chart, the maps φ_γ are homeomorphisms onto their image. Let $\mathcal{A} = \{\varphi_\gamma : \gamma \in \Gamma\}$. We claim that $\mathcal{A}$ is a smooth atlas. Indeed, since $\bigcup_{\gamma\in\Gamma} \gamma(U) = X$, the domains of the charts in $\mathcal{A}$ cover X. Furthermore, given φ_{γ_1} and φ_{γ_2},

$$\varphi_{\gamma_1} \circ {\varphi_{\gamma_2}}^{-1} = \varphi \circ (\gamma_1\gamma_2^{-1}) \circ \varphi^{-1},$$

which is C^∞ by assumption wherever defined. Hence, $\mathcal{A}$ is a smooth atlas. □

Problem 2. Let $\gamma : S^1 \to \mathbb{R}^2 \setminus \{0\}$ be a C^∞ loop. Show that for almost every $m \in \mathbb{R}$, the image of γ intersects the line $y = mx$ in at most finitely many points.

Solution 1. Let $\pi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{RP}^1$ denote the projection of a point x onto real projective space. We can view π as the composition of the projection of $\mathbb{R}^2 \setminus \{0\}$ onto S^1 by $x \mapsto x/\,||x||$, and the cover of $\mathbb{RP}^1$ by S^1 defined by identifying antipotal points. Crucially, the preimate of a point in $\mathbb{RP}^1$ is exactly a line in $\mathbb{R}^2$ passing through the origin, and any such line is obtained this way.

Notice that $f = \pi \circ \gamma$ is a map from a compact 1-manifold S^1 to a 1-manifold $\mathbb{RP}^1$. By Sard's theorem, the set of regular values has full measure. By the regular value theorem, the preimage of a point in $\mathbb{RP}^1$ is a 0-manifold. It must be compact because it is a subset of S^1, and is hence a finite set. Since the image of a finite set is finite, it follows that for any regular value of π, the intersection of the corresponding line and image of γ is finite. Finally, observe that on the set of non-vertical lines passing through, the map which associates its slope is a C^∞ chart, and hence takes sets of measure 0 to sets of measure 0. The result follows. □

Solution 2. Consider the family of maps $\gamma_m(t) = \begin{pmatrix} 1 & 0 \\ -m & 1 \end{pmatrix} \gamma(t) = (\gamma_1(t), \gamma_2(t) - m\gamma_1(t))$. Notice that the line $y = mx$ is the image of the line $y = 0$ under the map $\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$. Hence $\gamma(t)$ intersects the line $y = mx$ if and only if γ_m intersects the line $y = 0$. We will show that the map $F : \mathbb{R} \times S^1 \to \mathbb{R}^2 \setminus \{0\}$ defined by $F(m,t) = \gamma_m(t)$ is transvserse to the line $y = 0$. By Sard's transversality theorem, it will follow that for almost every m, γ_m is transverse to the line. By the regular value theorem, the preimage will be a compact 0-manifold, ie, a finite set. Since the image of a finite set is finite, the result will follow.

Now if $F(m,t)$ belongs the the line $y = 0$, it follows that $\gamma_1(t) \neq 0$ since $\gamma(t) \neq 0$. We compute

$$DF(m,t) = \begin{pmatrix} 0 & \gamma_1'(t) \\ -\gamma_1(t) & \gamma_2'(t) - m\gamma_1'(t) \end{pmatrix}$$

Since $\gamma_1(t) \neq 0$, the image of DF contains a multiple of e_2 (it is exactly the image of $\partial/\partial m$). Since the tangent bundle to the line $y = 0$ is spanned by e_1, it follow that the image of DF and the tangent bundle together span $\mathbb{R}^2$. Hence, F is transverse to $y = 0$. ☐

Solution 3 (embedded S^1 only). Define the map $F(m,x) = (x, mx)$, and $F_m(x) = F(m,x)$. Then the image of F_m is the line $y = mx$. Furthermore, $dF = \begin{pmatrix} 0 & x \\ 1 & m \end{pmatrix}$, so dF is an isomorphism except at 0. Since $0 \notin \mathrm{im}(\gamma)$, it follows that F is transverse to γ. By Sard's theorem for transversality, F_m is transverse to γ for almost every m. For each such m. the preimage of the image of γ under F_m is a finite collection of points, since it is a cmpact 0-manifold. Since the intersection of the image of γ and the line $y = mx$ is the image of this finite set, the image is finite for every such m. ☐

Problem 3. Consider a C^∞, k-dimensional foliation $\mathcal{F}$ on a C^∞ manifold M. Assume that there exists a flow ψ_t on M such that for every leaf L of $\mathcal{F}$ and $t \in \mathbb{R}$, $\psi_t(L)$ is also a leaf of $\mathcal{F}$. Furthermore, assume that if X is the generating vector field of ψ_t, then X is never tangent to the leaves of $\mathcal{F}$. Prove that the exists a $(k+1)$-dimensional foliation $\hat{\mathcal{F}}$ whose leaves contain the orbits of ψ_t and the leaves of $\mathcal{F}$.

Solution. We define a distribution E on M by

$$E(p) = T\mathcal{F}(p) \oplus \mathbb{R}X(p)$$

so that $\dim(E(p)) = k+1$ at every point (since $X(p) \notin T\mathcal{F}(p)$ for any p). We claim that E is involutive. Since $T\mathcal{F}$ is the tangent bundle of a foliation $\mathcal{F}$, it follows that $T\mathcal{F}$ is involutive. It therefore suffices to show that if Y is a vector field subordinate to $T\mathcal{F}$, then $[X,Y]$ is subordinate to E. We will show a stronger property that $[X,Y]$ is subordinate to $T\mathcal{F}$. To compute this bracket, we firstobserve that for any $p \in M$, $\varphi_t^Y(p)$ is contained in the leaf of p. Therefore, $\varphi_s^Y(\varphi_{-t}^X(p))$ is contained in the $\mathcal{F}$-leaf of $\varphi_{-t}^X(p)$ for all $s \in \mathbb{R}$. By the asssumption given in the problem, for every $t, s \in \mathbb{R}$, $\varphi_t^X \varphi_s^Y \varphi_{-t}^X(p)$ is contained in the $\mathcal{F}$-leaf of p. Hence, the derivative in s and t belongs to $T\mathcal{F}(p)$, as claimed.

Finally, by the Frobenius theorem, since E is involutive, it is integrable to a foliation. Since the orbits of ψ_t and leaves of $\mathcal{F}$ are both tangent to E, it follows that the new foliation contains both of them. ☐